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# Comonotonicity, Efficient Risk-sharing and Equilibria in markets with short-selling for concave law-invariant utilities

R.-A. Dana \*

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## Abstract

In finite markets with short-selling, conditions on agents' utilities insuring the existence of efficient allocations and equilibria are by now well understood. In infinite markets, a standard assumption is to assume that the individually rational utility set is compact. Its drawback is that one does not know whether this assumption holds except for very few examples as strictly risk averse expected utility maximizers with same priors. The contribution of the paper is to show that existence holds for the class of strictly concave second order stochastic dominance preserving utilities. In our setting, it coincides with the class of strictly concave law-invariant utilities. A key tool of the analysis is the domination result of Lansberger and Meilijson that states that attention may be restricted to comonotone allocations of aggregate risk. Efficient allocations are characterized as the solutions of utility weighted problems with weights expressed in terms of the asymptotic slopes of the restrictions of agents' utilities to constants. The class of utilities which is used is shown to be stable under aggregation.

**Keywords:** Law invariant utilities, comonotonicity, Pareto efficiency, equilibria with short-selling, aggregation, representative agent.

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# 1 Introduction

The problem of the existence of equilibria in finite markets with short-selling has first been considered in the early seventies by Grandmont [24], Hart [27] and Green [25] in the context of temporary equilibrium models and assets equilibrium models. It was later reconsidered by a number of authors (for a review of the subject in finite markets, see Allouch et al [3], Dana et al [17], Page [35]). Three sets of conditions were given for existence of an equilibrium:

- the assumption of existence of a no-arbitrage price, a price at which no investor could make costless unbounded utility nondecreasing purchases (see for example Grandmont [24], Hammond [26], Page [34], [38]) or equivalently under standard conditions on utilities, that aggregate demand exists at some price,
  - the no unbounded utility arbitrage condition, a condition of absence of collective arbitrage, which requires that investors do not engage in mutually compatible, utility nondecreasing trades (see for example, Hart [27], Page [34], Nielsen [33]),
  - and finally, the assumption that the individually rational utility set is compact (see for example, Dana et al [17], Nielsen [33], Page and Wooders [36]).
- Under suitable assumptions, these conditions were shown to be equivalent (see Dana et al [17] and Page and Wooders [36]).

While the problem of existence of an Arrow-Debreu equilibrium in infinite economies with consumption sets bounded below was considered as well understood at the end of the eighties (see Aliprantis, Brown and Burkinshaw [1] and Mas-Colell and Zame [32]), a number of papers discussed the difficulties raised by the issue of shortselling: Cheng [12], Brown-Werner [8], Dana and Le Van [14],[15], Dana et al [18] and Aliprantis et al [2], this list not being exhaustive. The finite dimension assumptions were shown not to be equivalent, the assumption of absence of free lunch or of absence of collective arbitrage too weak. The standard assumption has been to assume that the individually rational utility set is compact with the drawback that it is not known whether it is fulfilled except for very few examples (models with mean variance utilities or strictly risk averse expected utilities). For proving existence of equilibrium in infinite dimension economies with consumption sets unbounded below, most papers have used the topological version of Negishi's approach. Dana Le Van [14],[15] have used utility weights  $e$  and the excess utility correspondence. Their paper however relies on the assumption that the individually rational utility set is compact.

More recently, there has been renewed interest for the problem of existence and characterization of efficient allocations in markets with short-selling, in the mathematical finance literature. Indeed, for the last ten years, the problem of quantifying the risk of a financial position has been very popular in finance (see Föllmer and Schied [23] for an overview) and has led to the concept of *convex measure of risk*. Risk sharing of an aggregate capital between different units or different investors or of the risk of a bank between its subsidiaries, led to problems of efficiency with shortselling that have been mainly discussed in infinite dimension (see for example, Barrieu and El Karoui [6], Filipovic and Svindland [21] and Jouini et al [28] ). All of these papers have all considered law-invariant convex measures of risk.

To show existence of efficient allocations for law-invariant convex measures of risk, Filipovic and Svindland [21] and Jouini et al [28] have both used *the domination result* of Lansberger and Meilijson [30] that any allocation of an aggregate risk is dominated for second order stochastic dominance by a comonotone allocation. Comonotone allocations are allocations having the property that agents' wealths are non decreasing functions of aggregate wealth that add up to identity. They are said to fulfill a *mutuality principle*. Moreover, the wealths of any pair of agents are positively correlated. Since the early work of Borch [7], Arrow [4] and Wilson [39], they have played an important role in the theory of risk sharing between strictly concave expected utility maximizers, the efficient allocations of risk being comonotone. When utilities are second order stochastic dominance preserving, from the domination result, for efficiency issues, attention may be restricted to comonotone allocations. As comonotone allocations are almost compact, with mild continuity assumptions on utilities, the individually rational utility set is compact. When the state space is non atomic and the utilities are *concave*, the hypothesis that utilities are second order stochastic dominance preserving is equivalent to their law invariance. By definition, law invariant utility functions only depend on the distributions of wealths and include many standard utilities, the expected utility, the rank dependent expected utility, the prospect utility, Green and Jullien's utility ( see below), the opposite of a number of very well-known risk-measures used in finance as entropy or averagevar. They have been very popular in the decision theoretic literature of the eighties. However not all of them are concave. For example risk averse expected utilities are law invariant and concave while risk taker expected utilities are law invariant and convex.

The first aim of the paper is to show existence of efficient allocations and equilibria for markets with short-selling for concave second order stochastic

dominance preserving utilities that fulfill some mild continuity properties and are strictly concave for most of the agents. In view of unifying models used in economics and in finance, an  $l + m$  agents exchange economy is considered, the first  $l$  agents having monetary utilities (adding  $t$  units of cash to a position increases the utility of  $t$ ) and the last  $m$  agents having law invariant strictly concave utilities. State contingent claims are assumed to be in  $L^\infty$ , a choice that may seem odd given current financial markets and the horrors of  $(L^\infty)'$ . As the utilities that are considered in the paper have supergradients in  $L^1_+$ , attention may be restricted to countably additive prices (or prices in  $L^1_+$ ) and values of contingent claims are integrals with respect to pricing densities.

Since utility functions are concave, the utility weight version of Negishi's method may be used to show existence of efficient allocations and equilibria. Efficient allocations are characterized as the solutions of utility weighted problems for weights expressed in terms of the asymptotic slopes of the restrictions to constants of the agents' utilities. The same utility weights characterize the efficient sharings of a fixed amount of a non random wealth between  $l + m$  agents having as utilities on the reals the restrictions to reals of agents' utilities. The efficient utility weights are therefore defined as the solutions of a set of equalities (monetary agents have same weight, otherwise, they would exchange cash so as to increase aggregate utility) and strict inequalities expressed in terms of the asymptotic slopes of the restrictions to constants of the agents' utilities.

The second aim of the paper is to show that the class of utilities being studied is stable by aggregation. The aggregation problem is by no mean an easy problem. The strictly risk averse RDU class is not stable by aggregation while the class of Choquet integrals with respect to a convex distortion is stable. It is shown that the monetary agents have a representative agent with a monetary law invariant utility (the sup-convolution of the monetary agents' utilities). Strictly concave agents also have a representative agent but it depends on the efficient allocation considered (or on the set of utility weights characterizing the efficient allocation). At any efficient allocation, the representative agent of the whole economy has a law invariant strictly concave utility. Finally at any efficient allocations, the wealths of the strictly concave agents and the aggregate monetary wealth are comonotone. The anticomonicity of prices and aggregate risk is known to hold in some cases. The generality of the result remains an open question.

The paper is organized as follows. In section 2, the model is presented and some properties of law invariant, concave, norm continuous utilities

recalled. Using the domination result of Lansberger and Meilijson, the utility set is shown to be closed and the individually rational utility set is compact. Section 3 is devoted to the characterization of efficient allocations as solutions of utility weighted problems. A monetary representative agent independent of utility weights is introduced. Section 4 is devoted to existence of equilibria and to some of its qualitative and aggregation properties. An appendix contains the proof of the two main results of the paper.

## 2 The domination result for law invariant concave utilities

### 2.1 The model

We consider a standard Arrow-Debreu one good exchange economy under uncertainty with  $l + m$  agents. Agents trade the set of state-contingent claims and have homogeneous beliefs about states of the world. Given as primitive is a non-atomic probability space  $(\Omega, \mathcal{B}, P)$ , hence it supports a random variable  $U$  uniformly distributed on  $[0, 1]$ . Contingent claims are identified to elements of  $L^\infty(\Omega, \mathbb{R})$  that we now on write  $L^\infty$ . Agents are described by their endowments  $W_i \in L^\infty$ ,  $i = 1, \dots, l + m$  and their utilities. Let  $W := \sum_i W_i$  be the aggregate endowment with distribution function  $F_W$  assumed to be continuous. Agents' utilities,  $u_i : L^\infty \rightarrow \mathbb{R}$  are concave, monotone, law invariant (two random variables with same probability law, have same utility), continuous in the norm topology of  $L^\infty$ . We also assume that the utility of some agent fulfills the following continuity assumption that insures that the superdifferential of the utility is in  $L^1$  (see proposition 2 below).

**H**  $X_n \uparrow X$ : a.e. implies  $u(X_n) \uparrow u(X)$ .

A utility is monetary if it is monotone and fulfills

$$u_i(X + t) = u_i(X) + t \text{ for any } t \in \mathbb{R},$$

In other words, if the risk-free amount  $t \in \mathbb{R}$  is added to  $X$ , then the utility increases of  $t$ . The opposite of a concave monotone, monetary utility is called a *convex measure of risk*. Numerous examples of concave, monotone, law invariant monetary utilities may be found in Jouini et al [28] and in Föllmer and Schied [23].

We assume that the utilities of the first  $l$  agents are concave, monetary, law invariant and norm-continuous and that the utilities of the last  $m$  agents

are strictly concave. Let us give an example of such utilities.

### An example: Green and Jullien's utilities

Let  $X$  be a random variable and  $F_X(t) = P(X \leq t)$ ,  $t \in \mathbb{R}$  be its distribution function. The generalized inverse of  $F_X$  or quantile of  $X$  is defined by:

$$F_X^{-1}(0) = \text{essinf } X \text{ and } F_X^{-1}(t) = \inf\{z \in \mathbb{R} : F_X(z) \geq t\}, \text{ for all } t \in ]0, 1]$$

Let

$$u_L(X) := \int_0^1 L(t, F_X^{-1}(t)) dt, \text{ for all } X \in L^\infty \quad (1)$$

where  $L \in C^2([0, 1] \times \mathbb{R})$ ,  $\partial_x L \geq 0$ ,  $\partial_{xx} L < 0$  and  $\partial_{tx} L \leq 0$  on  $[0, 1] \times \mathbb{R}$ . The concavity of  $u_L$  is proven in Carlier and Dana [9]. The strict concavity follows from the representation formula

$$u_L(X) = \min_U \{E(L(U, X)) \mid F_X^{-1}(U) = X\}$$

where  $U$  is a uniform law and from the strict concavity of  $L$ . Moreover assumption **H** is fulfilled. Indeed, if  $X_n \uparrow X$ , then  $F_{X_n}^{-1} \uparrow F_X^{-1}$  and from the dominated convergence theorem,  $u_L(X_n) \uparrow u_L(X)$ .

In the case  $L(t, x) = f'(1 - t)U(x)$  with  $f$  convex  $C^1$  and  $U$  concave  $C^2$ , one obtains the rank dependent expected utility.

## 2.2 A few properties of concave law-invariant utilities

The aim of this section is to show that the utilities that are considered have two fundamental properties: they are second order stochastic dominance preserving and their superdifferential is in  $L^1$ .

We recall that  $X$  dominates  $Y$  for second order stochastic dominance (SSD), denoted  $X \succeq_2 Y$  if  $E(U(X)) \geq E(U(Y))$ , for every  $U : \mathbb{R} \rightarrow \mathbb{R}$  concave nondecreasing while  $X$  strictly dominates  $Y$  for SSD denoted  $X \succ_2 Y$  if  $E(U(X)) > E(U(Y))$  for every strictly concave nondecreasing utility function  $U$ . Hence  $X \sim_2 Y$  if and only if  $X$  and  $Y$  have same distribution. A map  $u : L^\infty \rightarrow \mathbb{R}$  (strictly) preserves SSD if  $X \succeq_2 Y$  ( $X \succ_2 Y$ ) implies  $u(X) \geq u(Y)$  ( $u(X) > u(Y)$ ). A map  $u : L^\infty \rightarrow \mathbb{R}$  is law invariant if two random variables with same distribution have same utility. Since  $X \sim_2 Y$  if and only if  $X$  and  $Y$  have same distribution, if  $u$  preserves SSD, then  $u$  is law invariant. The next proposition provides sufficient conditions for the converse to be true.

**Proposition 1** *Let  $(\Omega, \mathcal{B}, P)$  be non-atomic and  $u : L^\infty \rightarrow \mathbb{R}$  be concave,  $\|\cdot\|_\infty$  upper semi-continuous. Then*

1.  *$u$  is  $\sigma(L^\infty, L^1)$  upper semi-continuous,*
2.  *$u$  is SSD preserving if and only if  $u$  is law invariant and monotone.*

**Proof.** From proposition 4.1 and remark 4.4 in Jouini et al [29], a  $\|\cdot\|_\infty$ -closed convex law-invariant subset of  $L^\infty$  is  $\sigma(L^\infty, L^1)$  closed. Hence if  $u : L^\infty \rightarrow \mathbb{R}$  is concave and law-invariant,  $\|\cdot\|_\infty$  upper semi-continuous, then  $\{X \in L^\infty \mid u(X) \geq a\}$  is  $\|\cdot\|_\infty$ -closed convex law-invariant, hence is  $\sigma(L^\infty, L^1)$  closed and  $u$  is  $\sigma(L^\infty, L^1)$  upper semi-continuous. The second assertion follows from the first assertion and theorem 4.1 in Dana [13].  $\square$

Let us next recall the definition of a supergradient of  $u$  at  $X$ :

$$\partial u(X) = \{z \in (L^\infty)' \mid u(X) - u(X') \geq z \cdot (X - X'), \text{ for all } X' \in L^\infty\}$$

The norm continuity of  $u$  implies that supergradients exist while **H** insures that supergradients are in  $L_+^1$ .

**Proposition 2** *For any  $i$ ,  $\partial u_i(X) \neq \emptyset$  and convex. If  $u_i$  fulfills **H**, then  $\partial u_i(X) \subseteq L_+^1$  and is  $L^1$  closed.*

**Proof.** Since  $u_i$  is norm continuous, it follows from Aubin [5], p 108 that  $\partial u_i(X)$  is non empty, convex and  $\sigma((L^\infty)', L^\infty)$  compact. Since  $(L^\infty)'$  can be identified to the space of bounded finitely additive measures which vanish on  $P$  null sets, let  $Q \in \partial u_i(X)$ . Then  $Q$  is finitely-additive and non negative since  $u_i$  is monotone. Let us show that  $Q$  is  $\sigma$ -additive. Let  $X \in L^\infty$  be given and  $A_n \downarrow \emptyset$ . Then  $X_n = X - 1_{A_n} \uparrow X$ . We have

$$u_i(X) - u_i(X_n) \geq z \cdot (X - X_n) = Q(A_n) \geq 0$$

From **H**,  $u_i(X_n) \rightarrow u_i(X)$ , therefore  $Q(A_n) \rightarrow 0$ , which implies that  $Q$  is  $\sigma$ -additive.  $\square$

## 2.3 Comonotone allocations

Let

$$A(W) = \{(X_i)_{i=1}^{l+m} \in (L^\infty)^{l+m} \mid \sum_{i=1}^{l+m} X_i = W\}$$



be the set of feasible allocations for aggregate endowment  $W$ . We recall that  $(X_i)_{i=1}^{l+m} \in A(W)$  is (strictly) dominated by  $(X'_i)_{i=1}^{l+m} \in A(W)$  if  $u_i(X'_i) \geq u_i(X_i)$  for every  $i$  (with a strict inequality for some  $i$ ), while  $(X_i)_{i=1}^{l+m} \in A(W)$  is  $\succeq_2$  ( $\succ_2$ ) dominated by  $(X'_i)_{i=1}^{l+m} \in A(W)$  if  $X'_i \succeq_2 X_i$  for every  $i$  (strict for some  $i$ ). A  $\succeq_2$ -efficient allocation is a feasible allocation which is not strictly (not strictly  $\succeq_2$ ) dominated.

Comonotone feasible allocations play a crucial role in risk-sharing theory when utilities are concave monotone, law invariant and norm continuous. We recall that a pair of random variables  $(X, Y)$  is comonotone if there exists a subset  $B \subset \Omega \times \Omega$ ,  $P \otimes P(B) = 1$  such that

$$[X(s) - X(s')][Y(s) - Y(s')] \geq 0, \quad \forall (s, s') \in B$$

A family of random variables  $(X_i)_{i=1}^d$  is comonotone if any pair  $(X_i, X_j)$  is comonotone for all  $(i, j)$ . We next recall a number of useful results on comonotone allocations.

- Proposition 3**    1. An allocation  $(X_i)_{i=1}^{l+m} \in A(W)$  is comonotone if and only if there exists  $l + m$  non decreasing functions  $h_i$  on  $\mathbb{R}$  such that  $\sum_i h_i = Id$ , with  $X_i = h_i(W)$  a.e.
2. Any allocation in  $A(W)$  is  $\succeq_2$  dominated by a comonotone allocation in  $A(W)$ . If the allocation is not comonotone, then there exists an allocation that strictly dominates it.
3. A feasible allocation  $(X_i)_{i=1}^{l+m} \in A(W)$  is Pareto optimal iff there exist a set of strictly positive utility weights  $\lambda \in \mathbb{R}^{l+m}$  such that it is a solution to the problem

$$\mathcal{P}_\lambda(W) : \quad \sup \left\{ \sum_i \lambda_i u_i(X_i) \mid \sum_i X_i = W \right\}$$

Assertion 1 is well-known and proven in Denneberg [20]. It follows that the  $h_i$  are 1-Lipschitz. We refer to assertion 2 as the comonotone dominance result. Domination by comonotone allocations was originally proven by Landsberger and Meilijson [30] for two agents and an aggregate endowment supported by a finite set. Domination for two agents was extended by Filipovic and Swidland [21] to an aggregate risk in  $L^1$  by a limit argument. The  $n$  agents case is proven in Dana and Meilijson [19] and Ludovski and Rüschendorf [31] also by limits arguments. A direct proof with a constructive algorithm as well as strict dominance for non comonotone allocations is

proven in Carlier et al [10] for a non-atomic probability space. Assertion 3 with non negative utility weights is well-known. Utility weights have to be positive since if  $(X_i)_{i=1}^{l+m}$  is efficient and say  $\lambda_1 = 0$ , then agent 1 can give a strictly positive constant amount to any agent with strictly positive utility weight increasing the aggregate utility, contradicting efficiency of the allocation.

$$\text{Let } U(W) = \{v \in \mathbb{R}^{l+m} \mid v_i \leq u_i(X_i), \forall i, \text{ for some } (X_i)_{i=1}^{l+m} \in A(W)\}$$

be the utility set. For  $a \in \mathbb{R}^{l+m}$ , let

$$V(a, W) = \{v \in \mathbb{R}^{l+m} \mid a_i \leq v_i \leq u_i(X_i), \text{ for all } i, \text{ for some } (X_i)_{i=1}^{l+m} \in A(W)\}.$$

**Proposition 4** 1. For any  $W \in L^\infty$ ,  $U(W)$  is closed and convex,

2. For any  $W \in L^\infty$ ,  $a \in \mathbb{R}^{l+m}$ ,  $V(a, W)$  is compact. In other words, any subset of the utility set which is bounded below is bounded above.

**Proof.** Convexity is standard. To prove that  $U(W)$  is closed, let  $v^n \rightarrow v$ ,  $v^n \in U(W)$ . From proposition 3, for any  $i = 1, \dots, l+m$ , there exists a sequence  $X_i^n(W)$  with  $X_i^n$  non-decreasing 1-Lipschitz such that  $v_i^n \leq u_i(X_i^n(W))$  for all  $n$  and  $i$ . Since  $X_i^n$  is 1-Lipschitz, we have for all  $n$  and  $i$ , assuming w.l.o.g. that  $0 \in \text{support } W$

$$v_i^n \leq u_i(X_i^n(W)) \leq u_i(X_i^n(0) + \|W\|_\infty) \quad (2)$$

which implies that the sequence  $X_i^n(0)$ ,  $i = 1, \dots, l+m$  is bounded below. As  $\sum_i X_i^n(0) = 0$ , it is bounded above. Since  $X_i^n$  is 1-Lipschitz, it is uniformly bounded on compact subsets. From Ascoli's theorem, a subsequence converges uniformly to  $X_i$  non-decreasing 1-Lipschitz on the support of  $W$  and  $\|X_i^n(W) - X_i(W)\|_\infty \rightarrow 0$ . As  $u_i$  is norm-continuous, we have for any  $i$   $u_i(X_i) = \lim u_i(X_i^n) \geq v_i$ . Hence  $v \in U(W)$  proving that it is closed. The proof of the second assertion which uses similar arguments is omitted.  $\square$

It follows from the proof of proposition 4 that any sequence of comonotone allocations  $(X_i^n(W))_{i=1}^{l+m}$  with utilities bounded above or below has a subsequence converging to a comonotone allocation in the  $L^\infty$  norm.

Existence of an equilibrium with prices in  $(L^\infty)'$  follows from proposition 4 and for example from Brown and Werner [8]. If there are no monetary agents, then existence of an equilibrium with prices in  $L^1$  follows from Dana and Le Van [15]. By using the utility weight approach, efficient allocations can be fully characterized and the qualitative properties of equilibrium wealths and

prices analysed. Furthermore aggregation properties of the utilities considered may be discussed. This is therefore the route we follow.

We shall make extensive use of the following corollary:

**Corollary 1** *If  $l = 0$  or  $l = 1$  then any efficient allocation must be comonotone. Furthermore if  $\mathcal{P}_\lambda(\mathcal{W})$  has a solution, it is unique.*

**Proof.** From proposition 1, agents' utilities are SSD preserving. Let  $(X_i)_{i=1}^{l+m}$  be efficient and assume that the first agent has a concave utility while the others have strictly concave utilities. From assertion 2 of proposition 3, if  $(X_i)_{i=1}^{l+m}$  is not comonotone, it is strictly dominated for SSD by a feasible comonotone allocation  $(Y_i)_{i=1}^{l+m}$ . As we must have for some  $i > 2$ ,  $X_i \neq Y_i$  efficiency of  $(X_i)_{i=1}^{l+m}$  is contradicted. Hence a Pareto optimal allocation is comonotone. Suppose that  $\mathcal{P}_\lambda(\mathcal{W})$  has two solutions  $(X_i)_{i=1}^{l+m}$  and  $(Y_i)_{i=1}^{l+m}$ . Then we must have for some  $i > 2$ ,  $X_i \neq Y_i$ . Then  $(\frac{X_i+Y_i}{2})_{i=1}^{l+m}$  does strictly better, a contradiction.  $\square$

**Remark.** We let the reader verify that if all utilities except at most one, are strictly risk averse, then corollary 1 also holds true.

### 3 Efficient risk-sharing

This section contains one of the main result of the paper. As standard in Negishi's method, efficient allocations are characterized as the solutions of weighted utilities problems. However, the utility weights for which there are solutions is not known. Theorem 1 shows that the utility weights for which the weighted utility problems have solutions are those for which the sharing of one good between agents having as utilities the restrictions of the original utilities to the reals has a solution.

Consider the problem  $\mathcal{P}_\lambda(x)$  of optimal sharing of the amount  $x \in \mathbb{R}$  between the  $l + m$  agents with utilities  $u_i(x_i)$ ,  $x_i \in \mathbb{R}$ .

$$\mathcal{P}_\lambda(x) : \sup \sum_{i=1}^{l+m} \lambda_i u_i(x_i), \quad \sum_{i=1}^{l+m} x_i = x, \quad x_i, x \in \mathbb{R} \quad (3)$$

To simplify the analysis, we assume that  $u_i(x_i)$   $x_i \in \mathbb{R}$  is  $C^1$  for all  $i$ . Let  $u'_j(\infty) = \lim_{x \rightarrow \infty} u'_j(x)$ ,  $u'_j(-\infty) = \lim_{x \rightarrow -\infty} u'_j(x)$  be its positive and negative asymptotic slopes. For  $1 \leq i \leq l$ , as  $u_i(x) = x + b$  for all  $x \in \mathbb{R}$ ,  $u'_i(-\infty) = u'_i(\infty) = 1$ .

### 3.1 The main result

**Theorem 1** *The following are equivalent :*

1.  $\mathcal{P}_\lambda(W)$  has a solution for any  $W \in L^\infty$ ,
2.  $P_\lambda(x)$  has a solution for any  $x \in \mathbb{R}$ ,
3.  $\lambda \gg 0$  and

$$\text{for all } (i, j), i \neq j, 1 \leq i \leq l, 1 \leq j \leq l, \lambda_i = \lambda_j \quad (4)$$

$$\forall (i, j), i \neq j, l+1 \leq i \leq l+m, l+1 \leq j \leq l+m, \frac{\lambda_i}{\lambda_j} < \frac{u'_j(-\infty)}{u'_i(+\infty)} \quad (5)$$

$$\forall (i, j), 1 \leq i \leq l, l+1 \leq j \leq l+m, \lambda_j u'_j(\infty) < \lambda_i < \lambda_j u'_j(-\infty) \quad (6)$$

The proof of theorem 1 may be found in the appendix. Let  $u_0$  be the sup-convolution of the monetary utilities  $u_i$ ,  $1 \leq i \leq l$  defined by

$$u_0(X_0) = \sup \{u_1(X_1) + u_2(X_2) \dots + u_l(X_l), X_1 + \dots + X_l = X_0\} \quad (7)$$

Let

$$\lambda_0 = \lambda_i, 1 \leq i \leq l \quad \text{and} \quad \tilde{\lambda} = (\lambda_0, \lambda_{l+1}, \dots, \lambda_{l+m}) \quad (8)$$

Under (4), (5), (6), it follows from the proof of theorem 1 that solving problem  $\mathcal{P}_\lambda(W)$  is equivalent to solving an  $m+1$  agents risk-sharing problem of  $W$ ,  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$  where the last  $m$  agents have utilities  $u_i$  and the agent with index 0 has utility  $u_0$

$$\tilde{\mathcal{P}}_{\tilde{\lambda}}(W) : \sup \left\{ \lambda_0 u_0(X_0) + \sum_{i \geq l+1} \lambda_i u_i(X_i), X_0 + \sum_{i \geq l+1} X_i = W \right\} \quad (9)$$

and solving  $u_0$  at the solution  $\tilde{X}_0(\tilde{\lambda}, W)$  of  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$ . Equivalently, the  $l$  first agents aggregate into a monetary agent with utility  $u_0$  independent of the efficient allocation and that we are brought down to solve risk sharing problems between the representative monetary agent and the last  $m$  agents.

### 3.2 Properties of the value function and of the efficient risk sharing rule

In the remainder of this section, we focus on the value and the solutions of problem  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$ . Let  $H = \{0, l+1, \dots, l+m\}$ . Consider the open subset of  $\mathbb{R}_+^{m+1}$ ,

$$D = \{\tilde{\lambda} \in \mathbb{R}_+^{m+1} \mid 0 < \frac{\lambda_i}{\lambda_j} < \frac{u'_j(-\infty)}{u'_i(+\infty)}, \lambda_j u'_j(\infty) < \lambda_0 < \lambda_j u'_j(-\infty), i, j \in H\}$$

For any  $\tilde{\lambda} \in D$ ,  $W \in L^\infty$ , from theorem 1 and corollary 1,  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$  has a unique solution which is comonotone. Let  $h(\tilde{\lambda}, W)$ ,  $(\tilde{X}_i(\tilde{\lambda}, W))_{i \in H}$  and  $(\tilde{v}_i(\tilde{\lambda}, W))_{i \in H} = (u_i(\tilde{X}_i(\tilde{\lambda}, W)))_{i \in H}$  denote respectively the value function of  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$ , its solution and the optimal utilities.

**Proposition 5** 1. For any  $\tilde{\lambda} \in D$ ,  $h(\tilde{\lambda}, \cdot)$  is concave, monotone, law invariant, continuous in the norm topology of  $L^\infty$ .

2. For any  $W \in L^\infty$ ,  $h(\cdot, W)$  is convex and continuous on  $D$ .

**Proof.** In assertion one, the concavity and monotonicity of  $h(\tilde{\lambda}, \cdot)$  are obvious. Since  $\infty > h(\tilde{\lambda}, W) \geq \sum_{i=l+1}^{l+m} \lambda_i u_i(0) + \lambda_0 u_0(W) > -\infty$  for any  $W \in L^\infty$  and  $\tilde{\lambda} \in D$ , the domain of  $h(\tilde{\lambda}, \cdot)$  is  $L^\infty$  and  $h(\tilde{\lambda}, \cdot)$  is norm-continuous on its domain. To prove the law invariance of  $h(\tilde{\lambda}, \cdot)$ , we have  $h(\tilde{\lambda}, W) = \sup \sum_{i \in H} \lambda_i u_i(\tilde{X}_i(W))$  over comonotone allocations of  $W$ . If  $W'$  has same distribution as  $W$ , then  $u_i(\tilde{X}_i(W)) = u_i(\tilde{X}_i(W'))$  since the utilities  $u_i$  are law invariant, hence  $h(\tilde{\lambda}, \cdot)$  is law invariant. To prove assertion 2, the function  $h(\cdot, W)$  is convex as supremum of linear functions. As  $D \subseteq \text{dom } h(\cdot, W)$  and is open,  $h(\cdot, W)$  is continuous on  $D$ .  $\square$

Let

$$\partial_2 h(\tilde{\lambda}, W) = \{z \in (L^\infty)' \mid h(\tilde{\lambda}, W) - h(\tilde{\lambda}, W') \geq z \cdot (W - W'), \forall W' \in L^\infty\}$$

be the superdifferential of the value function of problem  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$ .

**Lemma 1** 1. For any  $\tilde{\lambda} \in D$  and  $W \in L^\infty$ ,  $\partial_2 h(\tilde{\lambda}, W) \neq \emptyset$  and convex.

2. For any  $\tilde{\lambda} \in D$  and any  $W \in L^\infty$

$$\partial_2 \tilde{h}(\tilde{\lambda}, W) = \cap_j \lambda_j \partial_2 u_j(\tilde{X}_j(\tilde{\lambda}, W)) \quad (10)$$

$$\partial_2 u_0(\tilde{X}_0(\tilde{\lambda}, W)) = \cap_{i=1}^l \partial_2 u_i(X_i(\tilde{\lambda}, W)) \quad (11)$$

for any allocation  $(X_i(\tilde{\lambda}, W))_{i=1}^l$  solving (7) at  $X_0(\tilde{\lambda}, W)$ .

3. If some agent fulfills **H**, then  $\partial_2 h(\tilde{\lambda}, W) \subseteq L_+^1$  and is  $L^1$  closed and convex.

**Proof.** As  $h(\tilde{\lambda}, \cdot)$  is norm-continuous for any  $\tilde{\lambda} \in D$ ,  $\partial_2 \tilde{h}(\tilde{\lambda}, W)$  is non empty and convex. The second assertion is standard. Assertion 3 follows from assertion 2 and lemma 1.  $\square$

**Proposition 6** 1. If  $\tilde{\lambda}^n \in D \rightarrow \tilde{\lambda} \in D$ , then  $\tilde{X}_i(\tilde{\lambda}^n, W) \rightarrow \tilde{X}_i(\tilde{\lambda}, W)$  in the norm topology of  $L^\infty$ .

2. If  $\tilde{\lambda}^n \in D \rightarrow \tilde{\lambda} \in \partial D$ , then, for some  $i$ ,  $\tilde{X}_i(\tilde{\lambda}^n, W) \rightarrow -\infty$  a.e. and  $\tilde{v}_i(\tilde{\lambda}, W) \rightarrow -\infty$

**Proof.** To prove the first assertion, as shown in the proof of 3 implies 1 in theorem 1, if a subsequence of  $\tilde{X}_i(\tilde{\lambda}^n, 0)$  is unbounded, then  $\sum_{i \in H} \lambda_i u_i(\tilde{X}_i(\tilde{\lambda}^n, W))$  is dominated for  $n$  large enough, hence  $\sum_{i \in H} \lambda_i^n u_i(\tilde{X}_i(\tilde{\lambda}^n, W))$  is dominated for  $n$  large enough contradicting the optimality of  $\tilde{X}_i(\tilde{\lambda}^n, W)$ . Therefore  $\tilde{X}_i(\tilde{\lambda}^n, 0)$  is bounded. As  $\tilde{X}_i(\tilde{\lambda}^n, \cdot)$  is 1-Lipschitz,  $\tilde{X}_i(\tilde{\lambda}^n, \cdot)$  is uniformly bounded on compact sets and  $\tilde{X}_i(\tilde{\lambda}^n, W)$  has a subsequence converging to  $\tilde{X}_i^*$  in the norm topology of  $L^\infty$ . We have for each  $n$ , for any comonotone allocation  $(\tilde{X}_i)$  of  $W$ ,

$$\sum_{i \in H} \lambda_i^n u_i(\tilde{X}_i(\tilde{\lambda}^n, W)) \geq \sum_{i \in H} \lambda_i^n u_i(\tilde{X}_i(W))$$

As  $u_i$  is norm continuous, taking the limit, we obtain that

$$\sum_{i \in H} \lambda_i u_i(\tilde{X}_i^*) \geq \sum_{i \in H} \lambda_i u_i(\tilde{X}_i(W))$$

As  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$  has a unique solution  $(\tilde{X}_i(\tilde{\lambda}, W))_{i \in H}$ ,  $\tilde{X}_i^*(W) = \tilde{X}_i(\tilde{\lambda}, W)$  for all  $i \in H$ . As  $\tilde{X}_i(\tilde{\lambda}^n, W)$  has a unique limit point in the norm topology, it converges to  $\tilde{X}_i(\tilde{\lambda}, W)$ .

To prove the second assertion, let  $\tilde{\lambda}^n \rightarrow \tilde{\lambda} \in \partial D$ . Suppose that  $u_i(\tilde{X}_i(\tilde{\lambda}^n, W))$  is bounded below for all  $i$ . From proposition 4, it is bounded above and  $\tilde{X}_i(\tilde{\lambda}^n, \cdot)$  is uniformly bounded on compact sets. From Ascoli's theorem,  $\tilde{X}_i(\tilde{\lambda}^n, W)$  has a subsequence converging in the norm topology. As in the previous proof, we obtain that the limit is the solution of  $\tilde{\mathcal{P}}_{\tilde{\lambda}}(W)$ . From theorem 1,  $\tilde{\lambda} \in D$  contradicting the assumption that  $\tilde{\lambda} \in \partial D$ . Hence for some  $i$ ,  $\tilde{v}_i(\tilde{\lambda}^n, W) \rightarrow -\infty$ . Since for every  $n$ ,  $\tilde{X}_i(\tilde{\lambda}^n, \cdot)$  is 1-Lipschitz, we have

$$\tilde{X}_i(\tilde{\lambda}^n, 0) - \|W\|_\infty \leq \tilde{X}_i(\tilde{\lambda}^n, W) \leq \tilde{X}_i(\tilde{\lambda}^n, 0) + \|W\|_\infty \quad (12)$$

As  $u_i$  is increasing, we obtain

$$u_i(\tilde{X}_i(\tilde{\lambda}^n, 0) - \|W\|_\infty) \leq \tilde{v}_i(\tilde{\lambda}^n, W) \leq u_i(\tilde{X}_i(\tilde{\lambda}^n, 0) + \|W\|_\infty)$$

Hence  $\tilde{X}_i(\tilde{\lambda}^n, 0) \rightarrow -\infty$ . From (12),  $\tilde{X}_i(\tilde{\lambda}^n, W) \rightarrow -\infty$  a.e. as was to be proven.  $\square$

## 4 Equilibria

From now on, we assume that **H** is fulfilled.

Let us first consider the original  $l + m$  agents economy. A feasible allocation and a price  $((X_i^*)_{i=1}^{l+m}, z^*) \in A(W) \times L_+^1$  is an equilibrium if for any  $i = 1, \dots, l + m$ ,  $X_i^*$  solves

$$\max u_i(X_i) \text{ s.t. } E(z^* X_i) \leq E(z^* W_i), \quad X_i \in L^\infty$$

### 4.1 The fictitious $m + 1$ agents' economy

We have created a fictitious  $m + 1$  agents economy. To define a concept of equilibrium for that economy, we assume that agent 0 is endowed with  $W_0 = \sum_{i=1}^l W_i$ . An allocation and a price  $((\tilde{X}_i^*), z^*) \in (L^\infty)^{m+1} \times L_+^1$  is an equilibrium if  $X_0^* + \sum_{i=l+1}^{l+m} \tilde{X}_i^* = \sum_{i=1}^{l+m} W_i$  and for any  $i \in H$ ,  $\tilde{X}_i^*$  solves

$$\max u_i(X_i) \text{ s.t. } E(z^* X_i) \leq E(z^* W_i), \quad X_i \in L^\infty$$

### 4.2 The excess utility correspondence

For  $\tilde{\lambda} \in D$ , the *excess utility* correspondence  $\tilde{E} : D \rightarrow \mathbb{R}^{m+1}$  is defined by

$$E_i(\tilde{\lambda}, W) = \left\{ \frac{z \cdot (\tilde{X}_i(\tilde{\lambda}, W) - W_i)}{\lambda_i}, \quad z \in \partial_2 h(\tilde{\lambda}, W) \right\}, \text{ for all } i \quad (13)$$

Restricting attention to agent  $i$ , from (10), we have that for all  $i$

$$\tilde{v}_i(\tilde{\lambda}, W) - u_i(W_i) \geq \frac{z \cdot (X_i(\tilde{\lambda}, W) - W_i)}{\lambda_i}, \quad z \in \partial_2 h(\tilde{\lambda}, W) \quad (14)$$

We next show that  $\tilde{E}$  has the properties of a finite dimensional excess demand correspondence.

**Proposition 7** *1. For any  $W \in L^\infty$ ,  $\tilde{\lambda} \in D$ ,  $\tilde{E}(\cdot, W)$  is a convex, compact, non empty valued, upper hemi-continuous correspondence, which satisfies Walras-law  $\tilde{\lambda} \cdot \tilde{E}(\tilde{\lambda}, W) = 0$ .*

*2. If  $\tilde{\lambda}^n \in D \rightarrow \tilde{\lambda} \in \partial D$ , then for some  $i$ ,  $\tilde{E}_i(\tilde{\lambda}^n, W) \rightarrow -\infty$ .*

**Proof.** The proof of assertion 1 is similar to that of proposition 3.5 in Dana and Le Van [15]. To prove the second assertion, let  $\tilde{\lambda}^n \in D \rightarrow \tilde{\lambda} \in \partial D$ . We then have for any  $t_i^n \in \tilde{E}_i(\tilde{\lambda}^n, W)$

$$\tilde{v}_i(\tilde{\lambda}^n, W) - u_i(W_i) \geq t_i^n$$

From proposition 6, for some  $j$ ,  $\tilde{v}_j(\tilde{\lambda}^n, W) \rightarrow -\infty$ , hence  $t_j^n \rightarrow -\infty$  as was to be proven.  $\square$

### 4.3 Existence of equilibrium

For sake of completeness, a generalization of the Gale-Nikkaido's lemma proven by Florenzano and Le Van [22] is first recalled. Given a subset  $C \subseteq \mathbb{R}^m$ ,  $C^0 = \{p \in \mathbb{R}^m \mid p \cdot X \leq 0, \text{ for all } X \in C\}$  is the polar of  $C$ .

**Lemma 2** *Let  $C$  be a closed convex cone in  $\mathbb{R}^m$  which is not a half-space. Let  $S$  denote the unit sphere of  $\mathbb{R}^m$ . Let  $Z$  be an upper-semi-continuous, non empty convex compact valued correspondence from  $C \cap S$  into  $\mathbb{R}^m$  such that*

$$\text{for all } \lambda \in C \cap S, \exists z \in Z(\lambda), z \cdot \lambda \leq 0.$$

*Then there exists  $\bar{\lambda} \in C \cap S$  such that  $Z(\bar{\lambda}) \cap C^\circ \neq \emptyset$ .*

**Theorem 2** *There exists an equilibrium where agents' wealths are comonotone. At any equilibrium, the wealths of the agents with strictly concave utilities and the aggregate monetary wealth are comonotone. At each equilibrium, the monetary agents have a representative agent with a monetary law invariant utility independent of the equilibrium. The whole economy has a representative agent with a law invariant, concave, norm continuous utility.*

The proof includes two steps. The equilibria of the fictitious  $m+1$  economy in which the monetary agent with utility  $u_0$  is endowed with the aggregate monetary endowment and the strictly concave agents are endowed with their initial endowments is first constructed. The proof requires the generalization of the Gale-Nikkaido's lemma quoted above unless the asymptotic slopes verify  $u'_i(\infty) = 0$ ,  $u'_i(-\infty) = \infty$  for all  $i$ , case in which  $D$  is the interior of the positive orthant of  $\mathbb{R}^{m+1}$  and the classical version may be used. This step determines the strictly concave agents' equilibrium wealths and the aggregate monetary equilibrium wealth and the equilibrium pricing density. We then select equilibrium monetary wealths for the first  $l$  agents from the efficient sharings of the monetary equilibrium wealth. This second step which is equivalent to determining the equilibria of a monetary economy does not require the use of a fixed point theorem.

At equilibrium, the representative monetary utility is  $u_0$ . It is independent of the equilibrium. The overall representative agent's utility is  $h(\bar{\lambda}, W)$



where  $\bar{\lambda}$  is an equilibrium weight. It depends on the equilibrium. From proposition 5, it is law invariant, concave, norm continuous and fulfills **U**. While it follows from theorem 1 that the set of utility weights for which problems characterizing efficiency have solutions is the same as if agents were expected utility maximizers with utility indices  $u_i(x)$ ,  $x \in \mathbb{R}$ ,  $i = 1, \dots, l+m$ , we want to emphasize that the computation of efficient allocations (and of equilibrium prices) is a hard problem which has only been addressed for specific utilities in a number of papers for monetary utilities (for example [28], [21]), by Carlier and Dana [9] for the case of two agents and Carlier and Lachapelle [11] for  $n$  agents for consumption models and strictly concave utilities. An open question is under which conditions on utilities, prices are anti-comonotone to aggregate risk as in a standard strictly concave expected utility model. From Carlier and Dana [9], this holds true if for example some agent has a utility of the type Green and Jullien, but we believe that they are more general utilities.

## 5 Appendix

### 5.1 Proof of theorem 1

**Proof.** 1 implies 2, since, from assertion 2 of proposition 3, to show existence of a solution, we may restrict attention to comonotone allocations. As a comonotone allocation of a constant is a vector of  $\mathbb{R}^{l+m}$ , we are brought down to consider problem  $P_\lambda(x)$ .

Let us next show that 2 implies 3. If problem  $P_\lambda(x)$  has a solution  $(x_i^*)_{i=1}^m$ , from the first order conditions, we have  $\lambda_i u'_i(x_i^*) = \lambda_j u'_j(x_i^*)$  for any pair of agents. This implies that for any pair  $(i, j)$  of monetary agents,  $\lambda_i = \lambda_j$ , for any pair  $(i, j)$  of risk averse agents  $\frac{\lambda_i}{\lambda_j} < \frac{u'_j(-\infty)}{u'_i(+\infty)}$  and any pair  $(i, j)$  of risk neutral-risk averse agents  $\lambda_j u'_j(+\infty) < \lambda_i < \lambda_j u'_j(-\infty)$ .

To prove that 3 implies 1, let  $X_0 = \sum_{i=1}^l X_i$ . As  $\lambda_i = \lambda_j$ , let

$$u_0(X_0) = \sup \{u_1(X_1) + u_2(X_2) \dots + u_l(X_l), X_1 + \dots + X_l = X_0\}$$

be the sup-convolution of the law-invariant monetary utilities  $u_i, i = 1, \dots, l$ . From Filipovic and Swidland [21],  $u_0$  is concave finite valued, exact, law invariant,  $\|\cdot\|_\infty$  continuous. When the weights fulfill (4), (5), (6) solving problem  $\mathcal{P}_\lambda(W)$  may be brought down to solving an  $m+1$  agents risk sharing problem of  $W$   $\hat{\mathcal{P}}_{\bar{\lambda}}(W)$  where the last  $m$  agents have the utilities  $u_i$  and the

agent with index 0 has a monetary utility  $u_0$ :

$$\tilde{\mathcal{P}}_\lambda(W) : \sup \left\{ \lambda_0 u_0(X_0) + \sum_{i \geq l+1} \lambda_i u_i(X_i), X_0 + \sum_{i \geq l+1} X_i = W \right\}$$

with  $\lambda_0 = \lambda_i$ ,  $1 \leq i \leq l$  and solving the sup-convolution at the solution  $X_0(\lambda, W)$  of  $\mathcal{P}_\lambda(W)$ . To show existence of a solution to  $\tilde{\mathcal{P}}_\lambda(W)$ , from assertion 2 of proposition 3, we may restrict attention to comonotone allocations of  $W$  and from corollary 1, the solution if it exists is unique. Let  $H = \{0, l+1, \dots, l+m\}$  and  $(\tilde{X}_i(W))_{i \in H}$  be an  $m+1$  comonotone allocations of  $W$ . We have for all  $i \in H$ :

$$\tilde{X}_i(0) - \|W\|_\infty \leq \tilde{X}_i(W) \leq X_i(0) + \|W\|_\infty \quad (15)$$

Let us first show that  $\tilde{\mathcal{P}}_\lambda(W)$  has a finite value. From assertion (5) and (6), there exists  $B > 0$ ,  $A > 0$ , such that

$$\min_i \lambda_i u'_i(-B) > \lambda_0 > \max_j \lambda_j u'_j(A) \quad (16)$$

Let  $Y_i = \tilde{X}_i(0) + \|W\|_\infty$  and  $I = \{i \in H \mid Y_i \leq 0\}$  and  $J = \{i \in H \mid Y_i > 0\}$ . W.l.o.g. assume that  $0 \in I$ . Note that from 15,  $Y_i \geq \tilde{X}_i(W)$  for all  $i \in H$ . From (16), we first have for  $i \in I$ ,

$$\lambda_i(u_i(-B) - u_i(Y_i)) > u'_i(-B)(B - Y_i)$$

hence

$$\lambda_i u_i(-B) + \lambda_i u'_i(-B)B + (\max_{i \in J} \lambda_j u'_j(A))Y_i \geq \lambda_i u_i(Y_i)$$

For  $j \in J$ , we similarly have

$$\lambda_j u_j(A) - \lambda_j u'_j(A)A + (\max_{j \in J} \lambda_j u'_j(A))Y_j \geq \lambda_j u_j(Y_j)$$

Summing over  $i, j$ , we thus have since  $\sum_i Y_i = \|W\|_\infty$ ,

$$\begin{aligned} & \sum_{i \in I} \lambda_i u_i(-B) + \sum_{i \in J} \lambda_i u_i(A) - \sum_{i \in J} \lambda_i u'_i(A)A + \sum_{i \in I} \lambda_i u'_i(-B)B \\ & (\max_{j \in J} \lambda_j u'_j(A))\|W\|_\infty \geq \sum_{i \in H} \lambda_i u_i(Y_i) \geq \sum_{i \in H} \lambda_i u_i(\tilde{X}_i(W)) \end{aligned}$$

Taking the supremum over partitions  $I, J$  of  $H$  on the left hand side and the supremum over comonotone allocations of  $W$ , we thus obtain that

$$\sup_{(X_i(W))} \sum_{i \in H} \lambda_i u_i(\tilde{X}_i(W)) < \infty$$

Let us now show existence of an optimal solution for  $\tilde{\mathcal{P}}_\lambda(W)$ . Let  $(\tilde{X}_i^n(W))_{i \in H}$  be a maximizing sequence of  $\tilde{\mathcal{P}}_\lambda(W)$ . If this sequence is such that for all  $i \in H$ ,  $\tilde{X}_i^n(0)$  is bounded, from (15),  $(\tilde{X}_i^n)$  is uniformly bounded on compact sets. From Ascoli's theorem, for every  $i$ ,  $(\tilde{X}_i^n)$  has a subsequence converging to  $(\tilde{X}_i)$  uniformly. Hence  $\tilde{X}_i^n(W) \rightarrow \tilde{X}_i(W)$  for the  $L^\infty$  norm and  $(\tilde{X}_i(W))$  solves  $\tilde{\mathcal{P}}_\lambda(W)$ .

Suppose now that there exists a subsequence such that for some  $i$ ,  $\tilde{X}_i^n(0) \rightarrow -\infty$ . Let  $I, J, K$  be such that  $\tilde{X}_i^n(0) \rightarrow -\infty$ ,  $i \in I$ ,  $\tilde{X}_i^n(0) \rightarrow +\infty$ ,  $j \in J$ ,  $\tilde{X}_j^n(0)$  is bounded,  $k \in K$ . From our assumption  $I \neq \emptyset$ . As  $\sum_{i \in H} \tilde{X}_i^n(0) = 0$ ,  $J \neq \emptyset$ . We further assume that  $K \neq \emptyset$  and that  $0 \in K$  (if not the proof that follows has to be slightly modified). Note that, as  $\sum_i \tilde{X}_i^n(0) = 0$  and  $\sum_{i \in K} \tilde{X}_i^n(0)$  is bounded,  $\sum_{I \cup J} \tilde{X}_i^n(0)$  is bounded. From (4) and (5), there exists  $\varepsilon > 0$ ,  $B > 0$  and  $A > 0$ , such that

$$\lambda_i u'_i(-B) > \lambda_0 > (\max_{j \in J} \lambda_j u'_j(A)) + \varepsilon, \text{ for all } i \in I \quad (17)$$

Let us show that for  $n$  large enough, the sequence  $\tilde{X}_i^n(W)$  is dominated. Let  $\text{card}(A)$  denote the cardinal of the set  $A$ . Let  $Y_0^n = \tilde{X}_0^n + \sum_{i \in I \cup J} \tilde{X}_i^n + B \text{card}(I) - A \text{card}(J)$ ,  $Y_i^n(W) = -B$  if  $i \in I$ ,  $Y_i^n(W) = A$ ,  $i \in J$ ,  $Y_i^n(W) = \tilde{X}_i^n(W)$  otherwise. The allocation  $(Y_i^n(W))_{i \in H}$  is feasible. From (15), we have with  $m_0 = -\|W\|_\infty \text{card}(I \cup J) + B \text{card}(I) - A \text{card}(J)$ ,

$$\lambda_0(u_0(Y_0^n(W)) - u_0(\tilde{X}_0^n(W))) \geq \lambda_0 \left( \sum_{i \in I \cup J} \tilde{X}_i^n(0) + m_0 \right) =: M_0 \quad (18)$$

For  $i \in I$ , we have from (15) and (17), with  $M = -B - \|W\|_\infty$

$$\begin{aligned} \lambda_i(u_i(-B) - u_i(\tilde{X}_i^n(W))) &\geq \lambda_i(u_i(-B) - u_i(\tilde{X}_i^n(0) + \|W\|_\infty)) > \\ \lambda_i u'_i(-B)(M - \tilde{X}_i^n(0)) &> (\max_{i \in J} \lambda_j u'_j(A) + \varepsilon)(M - \tilde{X}_i^n(0)) \end{aligned} \quad (19)$$

for  $n$  large enough since  $\tilde{X}_i^n(0) \rightarrow -\infty$  and  $M - \tilde{X}_i^n(0) > 0$ .

For  $j \in J$ , we have from (15), with  $M' = A - \|W\|_\infty$ ,

$$\begin{aligned} \lambda_j(u_j(A) - u_j(\tilde{X}_j^n(W))) &\geq \lambda_j(u_j(A) - u_j(\tilde{X}_j^n(0) + \|W\|_\infty)) \\ \lambda_j u'_j(A)(M' - \tilde{X}_j^n(0)) &\geq (\max_{j \in J} \lambda_j u'_j(A))(M' - \tilde{X}_j^n(0)) \end{aligned} \quad (20)$$

for  $n$  large enough since  $\tilde{X}_j^n(0) \rightarrow \infty$  and  $(M' - \tilde{X}_j^n(0)) < 0$ . Summing over the  $m + 1$  agents, we obtain that for some  $m \in \mathbb{R}$ ,

$$\sum_i \lambda_i(u_i(Y_i^n(W)) - u_i(\tilde{X}_i^n(W))) > \varepsilon \left( \sum_{i \in I} (M - \tilde{X}_i^n(0)) \right) + m$$

As for  $n$  large enough, the right hand side is arbitrarily large, this contradicts the definition of  $(\tilde{X}_i^n(W))$ . Hence  $(\tilde{X}_i^n(0))$  must be bounded, a case already studied and assertion 1 is fulfilled.  $\square$

## 5.2 Proof of theorem 2

**Proof.** Let

$$K_n = \left\{ \tilde{\lambda} \in \mathbb{R}_+^{m+1} \mid 1/n \leq \frac{\lambda_i}{\lambda_j} \leq \frac{u'_j(-\infty)}{u'_i(+\infty)} - 1/n \right\}$$

Then  $K_n$  is a closed convex cone and  $K_n \subseteq D$ . From proposition 7 and lemma 2 applied on  $K_n$  to the upper-hemicontinuous and compact valued correspondence  $-\tilde{E}$  which verifies Walras-law, for any  $n$ , there exists  $\tilde{\lambda}_n \in K_n$  and  $e_n \in \tilde{E}(\tilde{\lambda}_n)$  such that  $-\tilde{\lambda}_n \cdot e_n \leq 0$  for all  $\tilde{\lambda} \in K_n$ . Let  $\bar{\lambda}$  be a limit point of  $\tilde{\lambda}_n$ . There are two cases:

Case 1)  $\bar{\lambda} \in D$ . Then by proposition 7,  $e_n \rightarrow \bar{e} \in \tilde{E}(\bar{\lambda})$ . Since  $-\tilde{\lambda} \cdot e_n \leq 0$  for all  $\tilde{\lambda} \in K_n$ , we have  $-\tilde{\lambda} \cdot \bar{e} \leq 0$  for all  $\tilde{\lambda} \in D$ . Since  $D$  is open, we must have  $-\bar{\lambda} \cdot \bar{e} < 0$  if  $\bar{e} \neq 0$  which contradicts Walras-Law. Hence  $\bar{e} = 0$ . This means that there exists  $\bar{z} \in \partial_2 \tilde{h}(\bar{\lambda}, W)$  such that  $0 = E(\bar{z}(X_i(\bar{\lambda}, W) - W_i))$  for  $i = 0$  or  $i \geq l + 1$ . Furthermore, since  $\bar{z} \in \bar{\lambda}_i \partial_2 u_i(X_i(\bar{\lambda}, W))$ , for all  $X_i \in L^\infty$ , we have

$$\bar{\lambda}_i(u_i(X_i(\bar{\lambda}, W)) - u_i(X_i)) \geq E(\bar{z}(X_i(\bar{\lambda}, W) - X_i)) = E(\bar{z}(W_i - X_i))$$

thus  $E(\bar{z}X_i) \leq E(\bar{z}W_i)$  implies  $u_i(X_i) \leq u_i(X_i(\bar{\lambda}, W))$  for  $i = 0$  or  $i \geq l + 1$ . Hence  $((X_i(\bar{\lambda}, W)), \bar{z})$  is an equilibrium for the fictitious  $m + 1$  agents economy.

Case 2)  $\bar{\lambda} \in \partial D$  and  $\|e_n\| \rightarrow \infty$ . Since  $(e_n)_i + u_i(W_i) \leq v_i(\tilde{\lambda}_n, W)$  for all  $i$   $((e_n)_i + u_i(W_i)) \in \tilde{U}(W)$  for all  $n$  where  $\tilde{U}(W)$  the utility set of the  $m + 1$  agents economy is defined by.

$$\tilde{U}(W) = \left\{ v \in \mathbb{R}^{m+1} \mid v_i \leq u_i(X_i), \forall i \in H, \sum_{i \in H} X_i = W \right\}$$

Hence  $(\frac{(e_n)_i}{\|e_n\|}) \rightarrow \bar{t} \in \tilde{U}_\infty(W) - \{0\}$  the asymptotic cone of  $\tilde{U}(W)$ . Therefore  $\tilde{\lambda} \cdot \bar{t} \leq 0$  for any  $\tilde{\lambda} \in \bar{D}$ . Furthermore, since  $-\tilde{\lambda} \cdot e_n \leq 0$ , for all  $\tilde{\lambda} \in K_n$ , we obtain at the limit  $-\tilde{\lambda} \cdot \bar{t} \leq 0$ , for all  $\tilde{\lambda} \in D$ . We thus have  $\tilde{\lambda} \cdot \bar{t} = 0$  for all

$\tilde{\lambda} \in D$  which is impossible since  $D$  is open. Hence only case 1 is possible.

Let us now construct an equilibrium for the  $l + m$  agents economy. Let  $\bar{\lambda}_i = \bar{\lambda}_0$ ,  $i = 1, \dots, l$ . As an equilibrium allocation is efficient, from theorem 1,  $(X_i(\bar{\lambda}, W))_{i=1}^l$  must be an optimal solution of (7) at  $X_0(\bar{\lambda}, W)$  that fulfills  $E(\bar{z} X_i) = E(\bar{z} W_i)$  for all  $1 \leq i \leq l$ . Indeed from lemma 1,  $\bar{z} \in \lambda_i \partial u_i(X_i(\bar{\lambda}, W))$  for all  $1 \leq i \leq l$ , hence  $X_i(\bar{\lambda}, W)$  solves

$$\max u_i(X_i) \text{ s.t. } E(\bar{z} X_i) \leq E(\bar{z} W_i), \quad X_i \in L^\infty$$

Let us show that from the monetary assumption, the equilibrium allocation may be constructed without using a fixed point theorem. Let  $\bar{1} \in \mathbb{R}^l$  be such that  $(\bar{1})_i = 1$  for all  $i = 1, \dots, l$ . Since  $\bar{z} \in \lambda_0 \partial u_0(X_0(\bar{\lambda}, W))$  and  $u_0$  is cash invariant, we have

$$\lambda_0 u_0(X_0(\bar{\lambda}, W)) - \lambda_0 u_0(X_0(\bar{\lambda}, W) + 1) = -\lambda_0 \geq -E(\bar{z})$$

$$\lambda_0 u_0(X_0(\bar{\lambda}, W)) - \lambda_0 u_0(X_0(\bar{\lambda}, W) - 1) = \lambda_0 \geq E(\bar{z})$$

hence  $E(\bar{z}) = \lambda_0$ .

Let  $(X_i(\bar{\lambda}, W))$  be any solution of (7) at  $X_0(\bar{\lambda}, W)$ . We claim that the allocation  $(X_i(\bar{\lambda}, W) + E(\frac{\bar{z}}{\lambda_0}(W_i - X_i(\bar{\lambda}, W))))$  solves (7). Indeed

$$\begin{aligned} \sum_{i=1}^l u_i(X_i(\bar{\lambda}, W) + E(\frac{\bar{z}}{\lambda_0}(W_i - X_i(\bar{\lambda}, W)))) &= \\ \sum_{i=1}^l u_i(X_i(\bar{\lambda}, W)) + \sum_{i=1}^l E(\frac{\bar{z}}{\lambda_0}(W_i - X_i(\bar{\lambda}, W))) &= \sum_{i=1}^l u_i(X_i(\bar{\lambda}, W)) \end{aligned}$$

the first equality holds true since  $u_i$  is monetary, the second since  $X_0(\bar{\lambda}, W) = \sum_{i=1}^l X_i(\bar{\lambda}, W)$  and  $E(\bar{z} X_0(\bar{\lambda}, W)) = E(\bar{z} \sum_{i=1}^l W_i)$  ( $\bar{z}$  being an equilibrium price for the  $m + 1$  agents economy). We also have since  $E(\bar{z}) = \lambda_0$ ,

$$E(\bar{z}(X_i(\bar{\lambda}, W) + E(\frac{\bar{z}}{\lambda_0}(W_i - X_i(\bar{\lambda}, W)))) = E(\bar{z} W_i) \quad \text{for all } 1 \leq i \leq l$$

Finally we claim that  $(X_i(\bar{\lambda}, W))_{i=1}^{l+m}, \bar{z}$  is an equilibrium of the  $l + m$  agents economy. Indeed markets clear ( $\sum_i X_i(\bar{\lambda}, W) = \sum_i W_i$ ) and for all  $i$ , we have that  $E(\bar{z} X_i) \leq E(\bar{z} W_i)$  implies  $u_i(X_i) \leq u_i(X_i(\bar{\lambda}, W))$ .  $\square$

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